Examination on
Basics of Quantum Theory



Version 1

This document contains exercise questions adapted from a selection of introductory materials in quantum mechanics taught to undergraduate students of physics with no prior training in quantum theory. It is intended to be presented as a test for online interlocutors to prove their grasp of the basics of quantum physics, should they claim any. The intended completion time is seven days beginning with this document's publication. Unless otherwise stated, the grade is computed as the fraction of all scored points out of all available points excepting questions marked as bonus. Unless otherwise stated, the passing grade shall be 60 % of non-bonus total.

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1. Wave Function Fundamentals

Suppose a particle is trapped in a one-dimensional interval $0 < x < a$ with the wave function

$$\psi(x,t) = N \sin\left(\frac{\pi x}{a}\right) e^{-\frac{\mathrm{i}}{\hbar}Et} \quad .$$

(a) Find all possible values of N such that $\|\psi\|^2 = 1$.
(4 points)

(b) Find the value of $\langle\hat{x}\rangle$ for this ψ.
(5 points)

(c) Using Schrödinger's equation (for a free particle or the infinite potential well for the given domain), find the value of E.
(5 points)

(d) (Bonus) Find the (normalized) momentum representation $\tilde{\psi}(p,t)$ of the above wave function.
(5 points)

Note: $\int \sin^2(bx)\,\mathrm{d}x = \frac{x}{2} - \frac{1}{4b}\sin(2bx)$.

2. Uncertainty Principle

Let $\hat{A}$ and $\hat{B}$ be arbitrary hermitean operators.

(a) Show that

$$\Delta A \cdot \Delta B \geq \frac{1}{2}\left|\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right| \quad ,$$

where $\Delta V := \sqrt{\left\langle\left(\hat{V} - \left\langle\hat{V}\right\rangle\right)^2\right\rangle}$ for any operator $\hat{V}$.
(6 points)

Note: It may help to define $\hat{Z} := -\mathrm{i}\left[\hat{A},\hat{B}\right]$. Consider a special case where $\left\langle\hat{A}\right\rangle = \left\langle\hat{B}\right\rangle = 0$. Show that this may be assumed without loss of generality.

(b) Using $\hat{x} \equiv x$ and $\hat{p} \equiv -\mathrm{i}\hbar\frac{\partial}{\partial x}$, and the inequality above, derive Heisenberg's original uncertainty relation

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

for these hermitean operators.
(2 points)

3. Quantum Harmonic Oscillator

All solutions $|\psi\rangle$ to the quantum mechanical harmonic oscillator problem

$$\mathrm{i}\hbar \frac{\partial}{\partial t} |\psi\rangle = \left[\frac{\hat{p}^2}{2m} + \frac{m}{2}\omega^2 \hat{x}^2 \right] |\psi\rangle$$

can be constructed from a countably infinite orthonormal basis $\left\{ e^{-\frac{\mathrm{i}}{\hbar}E_n t} |\varphi_n\rangle \right\}_{n \in \mathbb{N}_0}$, where $|\varphi_n\rangle$ are eigenstates of the given $\hat{H}$. The ladder operators

$$\hat{b} := \frac{\hat{p} - \mathrm{i}m\omega\,\hat{x}}{\sqrt{2m\hbar\omega}} \qquad\qquad \hat{b}^\dagger := \frac{\hat{p} + \mathrm{i}m\omega\,\hat{x}}{\sqrt{2m\hbar\omega}}$$

allow navigating between them:

$$\hat{b}\,|\varphi_n\rangle = \sqrt{n}\,|\varphi_{n-1}\rangle \qquad\qquad \hat{b}^\dagger\,|\varphi_n\rangle = \sqrt{n+1}\,|\varphi_{n+1}\rangle$$

(a) Compute a representation of $\hat{x}$ and $\hat{p}$ in terms of $\hat{b}$ and $\hat{b}^\dagger$. Use your result to evaluate $[\hat{x}, \hat{p}]$ using the properties of said ladder operators.

(5 points)

(b) Find the expected values of the hermitean operators $\hat{x}$, $\hat{p}$, $\hat{x}^2$, and $\hat{p}^2$ for all $|\varphi_n\rangle$. How could we have predicted the former two in advance?

(6 points)

(c) Find the value of $\Delta x \cdot \Delta p$ for all $|\varphi_n\rangle$. Is your result consistent with what you found in part (a) of this problem, in light of the theorem presented in Question 2?

(2 points)

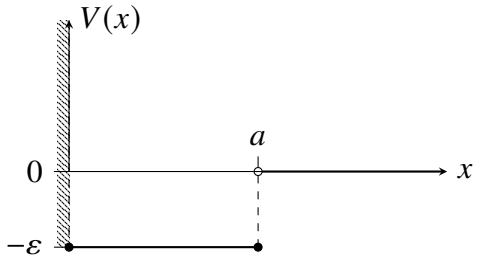
(d) Find the values of E_n corresponding to each $|\varphi_n\rangle$. What do you find noteworthy about the lowest possible one?

(5 points)

4. Solving Schrödinger's Equation

To the right is pictured for some $\varepsilon > 0$ a plot of the piecewise constant potential $V(x)$ given by

$$V(x) = \begin{cases} \infty & , \quad x < 0 \\ -\varepsilon & , \quad 0 \le x \le a \\ 0 & , \quad a < x \end{cases} .$$



(a) Find all stationary state functions $\varphi : \mathbb{R} \to \mathbb{C}$ with $\|\varphi\|^2 = 1$ that satisfy the time-independent Schrödinger equation $E\varphi(x) = \frac{\hat{p}^2}{2m}\varphi(x) + V(x)\varphi(x)$, with $\hat{p} \equiv -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}$ and $-\varepsilon < E < 0$.

(12 points)

(b) Which conditions cannot be met for $E < -\varepsilon$ and $E > 0$, respectively?

(4 points)

(c) (Bonus) There exists a transcendental equation restricting which values of E are possible. Derive a form of it.

(2 points)

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required score to pass: 34 out of 63 available points